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The effect of a finite anchoring energy on the transient periodic structures in nematic liquid crystals

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Non-equilibrium transient periodic structures have been widely observed in thermotropic and lyotropic nematic liquid crystals. So far only the case of strongly anchored nematics has been considered. Here we investigate the influence of a finite anchoring energy on the non-equilibrium transient pattern in a twist geometry. Both the domain wavelength and the threshold field for the non-equilibrium pattern have been calculated for different values of the anchoring energy.

1. Introduction

Recently many experimental and theoretical results about non-equilibrium transient periodic structures induced by magnetic fields in nematic liquid crystal layers have been reported [1–9]. The easy experimental investigation of these patterns makes them very interesting to study the general phenomenon of the occurrence of spontaneous dynamic structures [10–12]. All previously published papers consider the special case of strong director anchoring at the two plane interfaces of the nematic layer. In this paper we investigate the influence of a finite anchoring energy on the characteristic wavelength and on the threshold field for the periodic dynamic pattern. The following linear stability analysis is similar to that presented by Guyon *et al.* [2] and by Londberg *et al.* [6].

We consider a nematic layer of thickness d sandwiched between two parallel solid plates treated in such a way as to induce a homogeneous planar alignment of the director $\mathbf{n}$ (average molecular axis) along the x axis in the plane of the layer. A uniform magnetic field $\mathbf{B}$ can be applied along the y axis orthogonal to the x axis in the plane of the layer. The nematic is assumed to have a positive diamagnetic anisotropy $\chi_\alpha = \chi_\parallel - \chi_\perp$. As is well known [13, 14], the nematic layer exhibits an ordinary Freederickz transition to a homogeneous twisted director distortion when the magnetic field exceeds a threshold value B'_C. For strong anchoring the threshold field is given by

$$B'_C \equiv B_C = \pi/d\,(K_{22}/\chi_\alpha)^{1/2},$$

where K_{22} is the twist elastic constant of the nematic. A transition to a transient periodic pattern occurs over all the nematic layer if the magnetic field exceeds a new threshold value $B''_C > B'_C$. Among possible periodic patterns, the distortion which corresponds to the fastest growth rate suppresses all slower ones and becomes macroscopically observable. The decay time of the non-equilibrium periodic pattern greatly depends on the viscosity coefficients of the nematic and so it becomes very long in the case of polymeric and lyotropic nematics. This property makes these system

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particularly suitable for direct experimental investigations. The physical mechanism responsible for the onset of the non-equilibrium periodic texture has been widely discussed in [1–9]. The main reason for this behaviour is due to the strong coupling between the hydrodynamic motion and the director orientation for a periodic distortion. For large enough magnetic fields, this coupling reduces the effective orientational viscosity seen by the periodic distortion and favours its occurrence.

2. Theory of transient patterns in the presence of finite anchoring

In order to investigate the occurrence of the periodic pattern in the presence of a finite anchoring of the director on the two plane interfaces of the nematic layer we make the following standard assumptions:

(1) Reorientation of the director occurs entirely in the xy plane of the layer. Therefore the director can be written as $\mathbf{n} \equiv (\cos\theta, \sin\theta, 0)$, where θ is the angle which the director makes with the easy axis x.
(2) The director field is homogeneous along the y axis parallel to the magnetic field and thus, $\theta \equiv \theta(x, z, t)$. Furthermore we suppose that the hydrodynamic velocity is fully directed along the y axis: $\mathbf{v} \equiv (v_x = 0,\ v_y = v_y(x, z, t),\ v_z = 0)$.
(3) Since we are interested in the behaviour of the transient periodic pattern near the threshold we consider small values of θ and v_y. Therefore, we can use for the anchoring energy of the director the simple form $W(\theta) = W_0\theta^2$, where W_0 is the azimuthal anchoring energy coefficient ($\theta = 0$ corresponds to the easy director axis).

The linearized nematodynamic equations for the balance of bulk forces and torques are, respectively:

$$\rho \frac{\partial}{\partial t} v_y = \alpha_2 \frac{\partial^2 \theta}{\partial x \partial t} + \eta_c \frac{\partial^2 v_y}{\partial x^2} + \eta_a \frac{\partial^2 v_y}{\partial z^2} \tag{1}$$

and

$$\gamma_1 \frac{\partial}{\partial t} \theta = -\alpha_2 \frac{\partial v_y}{\partial x} + K_{33} \frac{\partial^2 \theta}{\partial x^2} + K_{22} \frac{\partial^2 \theta}{\partial z^2} + \chi_a B^2 \theta, \tag{2}$$

with the following boundary conditions on the surfaces ($\pm d/2$) of the two solid plates:

$$v_y(x, \pm d/2, t) = 0 \tag{3}$$

and

$$\pm K_{22} \left.\frac{\partial \theta(x, z, t)}{\partial z}\right|_{z = \pm d/2} = 2W_0 \theta(x, \pm d/2, t), \tag{4}$$

where the sign $\pm$ corresponds to the upper and lower surface of the nematic layer, respectively, and

$$\eta_a = \alpha_4/2, \quad \eta_c = (\alpha_4 + \alpha_5 - \alpha_6)/2 \quad \text{and} \quad \alpha_2 = (\gamma_2 - \gamma_1)/2$$

are viscosity coefficients [13] of the nematic; K_{33} and K_{22} are the bend and twist elastic constants, whilst W_0 is the anchoring energy coefficient [13].

As shown in [2] the contribution of the inertial term ($\rho(\partial v_y/\partial t)$ in equation (2)) can be neglected for small enough thicknesses of the nematic layer. Equation (1) to (4) can be rewritten in adimensional units by defining the time unit $\tau = \gamma_1/\chi_a B_c^2$ and the length unit d, where B_c is the Freederickz threshold field.

$$0 = \frac{\alpha_2}{\eta_a}\frac{\partial^2\theta}{\partial x\partial t} + \frac{\eta_c}{\eta_a}\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial z^2}, \tag{5}$$

$$\frac{\partial}{\partial t}\theta = -\frac{\alpha_2\partial v_y}{\gamma_1\partial x} + \frac{K_{33}}{\pi^2 K_{22}}\frac{\partial^2\theta}{\partial x^2} + \frac{1}{\pi^2}\frac{\partial^2\theta}{\partial z^2} + h^2\theta, \tag{6}$$

$$v_y(x, \pm 1/2, t) = 0, \tag{7}$$

$$\pm b\,\frac{\partial\theta(x,z,t)}{\partial z}\bigg|_{z=\pm\frac{1}{2}} = \theta(x, \pm 1/2, t), \tag{8}$$

where we have introduced the adimensional reduced magnetic field $h \equiv B/B_c$ and the adimensional extrapolation length [13, 14], $b \equiv K_{22}/(2W_0 d)$. We note that exactly the same equations are obtained for a nematic liquid crystal having a negative diamagnetic anisotropy. This is, for instance, the case of some lyotropic nematic liquid crystals. In this latter case, however, the magnetic field must be oriented parallel to the easy axis for the director (x axis) and the threshold field for the Freederickz transition is $B_c = \pi/d(K_{22}/|\chi_a|)^{1/2}$. We look, now, for a solution of equations (5) to (8) of the form

$$\theta(x,z,t) = \theta(z)\exp(iqx + st), \tag{9}$$

$$v_y(x,z,t) = v(z)\exp(iqx + st), \tag{10}$$

where q is the adimensional wavevector of the periodic pattern, and s is the adimensional growth rate of the periodic pattern. The initial homogeneous alignment of the director field becomes unstable with respect to the space fluctuation of wavevector q if the real part of s becomes greater than zero. By substituting equations (9) and (10) in equations (5)–(8) we obtain

$$\frac{d^2}{dz^2}v(z) = \frac{\eta_c}{\eta_a}q^2 v(z) - i\,\frac{\alpha_2 sq}{\eta_a}\,\theta(z), \tag{11}$$

$$\frac{d^2}{dz^2}\theta(z) = \left[\pi^2\left(s - h^2 + \frac{K_{33}q^2}{\pi^2 K_{22}}\right)\right]\theta(z) + i\,\frac{\pi^2\alpha_2 q}{\gamma_1}\,v(z), \tag{12}$$

$$v(\pm 1/2) = 0, \tag{13}$$

$$\pm b\,\frac{d}{dz}\theta(z)|_{z=\pm\frac{1}{2}} = \theta(\pm\tfrac{1}{2}). \tag{14}$$

Therefore $v(z)$ satisfies the fourth order differential equation

$$\frac{d^4}{dz^4}v(z) + \alpha\,\frac{d^2}{dz^2}v(z) + \beta v(z) = 0, \tag{15}$$

where

$$\alpha = \pi^2(h^2 - s) - \left(\frac{\eta_c}{\eta_a} + \frac{K_{33}}{K_{22}}\right)q^2 \tag{16}$$

and

$$\beta = \frac{K_{33}\eta_c}{K_{22}\eta_a}q^4 + \frac{\pi^2}{\eta_a}\left[\eta_c(s - h^2) - \frac{\alpha_2^2 s}{\gamma_1}\right]q^2. \tag{17}$$

The non-trivial solutions of equations (15) and (11) which satisfy the boundary conditions (13) and (14) are

$$v(z) = A\left[\frac{\exp(\lambda_1 z) + \exp(-\lambda_1 z)}{\exp(\lambda_1/2) + \exp(-\lambda_1/2)} - \frac{\exp(\lambda_2 z) + \exp(-\lambda_2 z)}{\exp(\lambda_2/2) + \exp(-\lambda_2/2)}\right], \quad (18)$$

$$\theta(z) = \frac{iA}{\alpha_2 qs}\left[(\eta_a\lambda_1^2 - \eta_c q^2)\frac{\exp(\lambda_1 z) + \exp(-\lambda_1 z)}{\exp(\lambda_1/2) + \exp(-\lambda_1/2)}\right]$$
$$- \left[(\eta_a\lambda_2^2 - \eta_c q^2)\frac{\exp(\lambda_2 z) + \exp(-\lambda_2 z)}{\exp(\lambda_2/2) + \exp(-\lambda_2/2)}\right], \quad (19)$$

where

$$\lambda_{1,2} = \pm\left[-\frac{\alpha}{2} \pm \left(\frac{\alpha^2}{4} - \beta\right)^{1/2}\right]^{1/2} \quad (20)$$

and A is an arbitrary adimensional constant which depends on the initial conditions. The growth rate s can be obtained by substituting the expressions for $v(z)$ and $\theta(z)$ in equations (11) and (12) to give

$$s = \frac{h^2 + (\lambda_{1,2}^2/\pi^2) - (K_{33}/\pi^2 K_{22})q^2}{1 + (\alpha_2^2 q^2)/[\gamma_1(\eta_a\lambda_{1,2}^2 - \eta_c q^2)]}. \quad (21)$$

By equating the two expressions of s corresponding to λ_1 and λ_2 in equation (21) we find

$$\lambda_1^2 = \frac{[-\alpha_2^2(\lambda_2^2 + \pi^2 h^2) + \gamma_1\eta_c\lambda_2^2]q^2 + [(K_{33}/K_{22})\alpha_2^2 + (\alpha_2^2 - \gamma_1\eta_c)(\eta_c/\eta_a)]q^4}{\eta_a\gamma_1\lambda_2^2 + (\alpha_2^2 - \gamma_1\eta_c)q^2}. \quad (22)$$

This equation relates λ_1 to λ_2 for a given value of the wavevector q and of the reduced magnetic field, h. Another relation concerning the same parameters can be obtained by substituting equations (18) and (19) into the boundary condition of the equation (14)

$$\frac{\eta_a}{b}(\lambda_1^2 - \lambda_2^2) = -(\eta_a\lambda_1^2 - \eta_c q^2)\lambda_1\frac{\exp(\lambda_1/2) - \exp(-\lambda_1/2)}{\exp(\lambda_1/2) + \exp(-\lambda_1/2)}$$
$$+ (\eta_a\lambda_2^2 - \eta_c q^2)\lambda_2\frac{\exp(\lambda_2/2) - \exp(-\lambda_2/2)}{\exp(\lambda_2/2) + \exp(-\lambda_2/2)}. \quad (23)$$

Equations (22) and (23) allow us to obtain λ_1 and λ_2 for different values of the wavevector q of the periodic pattern, of the reduced magnetic field h and of the anchoring energy coefficient ($b = K_{22}/2W_0 d$).

3. Threshold field of the periodic transient pattern

By substituting the solution of equations (22) and (23) for λ_1 or λ_2 into equation (21) we obtain the growth rate s of the periodic pattern of wavevector q. The threshold field for the growing of the periodic pattern is reached when the real part of the corresponding value of s becomes greater than zero. According to previous work [1–9] among various unstable patterns ($\mathrm{Re}\, s > 0$) the macroscopic one is that which exhibits the maximum value of $\mathrm{Re}\, s$. We denote by q_c the wavevector corresponding to this easy pattern. Equation (22) and (23) can be solved numerically to calculate $s(q)$

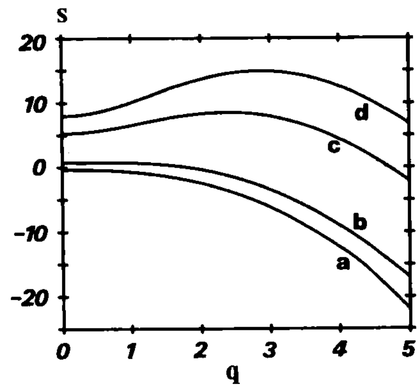


Figure 1. Adimensional growth rate s (time unit $\tau = \gamma_1/\chi_a B_c^2$) versus the adimensional wavevector q (length unit d = thickness of the nematic layer) of the periodic perturbation for the reduced extrapolation length $b = 0{\cdot}01$ ($b \equiv K_{22}/(2W_0 d)$) and for some values of the reduced magnetic field $h = B/B_c$. Curve a: $h = 0{\cdot}8$, curve b: $h = 1{\cdot}313$ ($h \approx h_c''$), curve c: $h = 2{\cdot}5$ and curve d: $h = 3$. The elastic and viscosity coefficients used for the numerical calculation are: $\eta_a = \alpha_4/2 = 0{\cdot}24\,\mathrm{p}$, $\eta_c = (\alpha_4 + \alpha_5 - \alpha_6)/2 = 1{\cdot}03\,\mathrm{p}$ and $\alpha_2 = (\gamma_2 - \gamma_1)/2 = -0{\cdot}77\,\mathrm{p}$, $\gamma_1 = 0{\cdot}76\,\mathrm{p}$, $K_{33} = 8 \times 10^{-7}$ dyn and $K_{22} = 3{\cdot}4 \times 10^{-7}$ dyn.

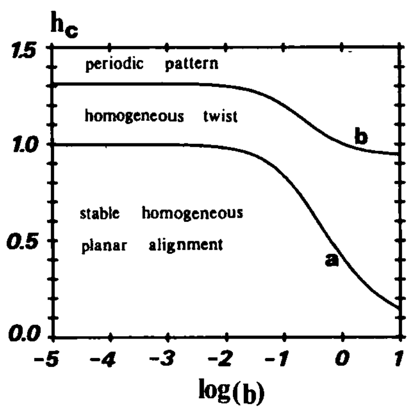
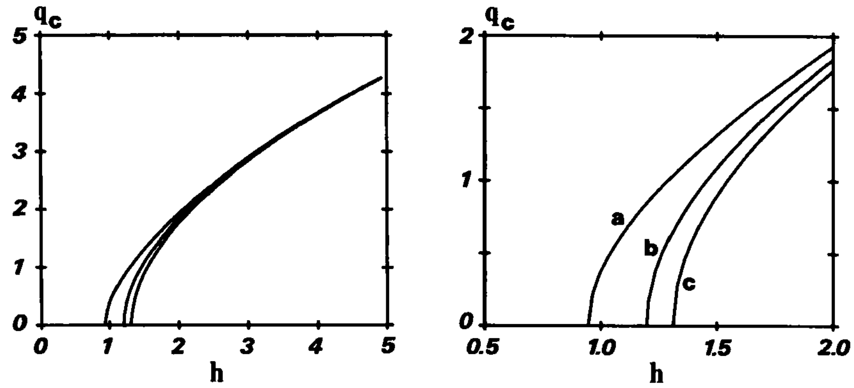


Figure 2. Threshold reduced fields h_c' (a curve) and h_c'' (b curve) versus the logarithm of the reduced extrapolation length b. h_c' corresponds to the threshold field for the ordinary Freederickz transition (see equation (26)) whilst h_c' corresponds to the threshold for the periodic transient pattern (see equation (31)). The numerical values of the elastic and viscosity coefficients of the nematic used to make the numerical calculations are the same as those in figure 1.

and the easy wavevector q_c for given values of the physical parameters of the nematic (see the values given in figure 1). As a main result of the numerical calculation we find that $s(q)$ always remains a real number and thus, a stationary transition to the periodic pattern occurs. Figure 1 shows some $s(q)$ curves for different values of the reduced magnetic field h and for $b = 0{\cdot}01$. Analogous behaviour is obtained at different values of the extrapolation length b. Depending on the value of the reduced magnetic field h and of the reduced extrapolation length b three different types of behaviour occur: (a) if $h < h_c'$, $s(q)$ is lower than zero for all values of q and thus, the planar homogeneous state is stable (see figure 1, curve a). (b) If $h_c' < h < h_c''$, the $s(q)$ curve



(*a*) (*b*)

Figure 3. Characteristic adimensional wavevector q_c of the periodic pattern as a function of the reduced magnetic field for some values of the reduced extrapolation length: (*a*) $b = 10$, (*b*) $b = 0{\cdot}1$, (*c*) $b = 0{\cdot}001$. In (*b*) details of the near threshold behaviour are evidenced. The numerical values of the elastic and viscosity coefficients of the nematic used to make the numerical calculations are the same as those in figure 1.

exhibits a positive maximum at $q = 0$ and thus, the ordinary homogeneous Freederickz transition occurs. (c) If $h > h_c''$, the relative maximum at $q = 0$ disappears and is replaced by a new maximum at a given wavevector $q = q_c \neq 0$. Therefore $h = h_c'$ corresponds to the ordinary threshold field of the Freederickz transition whilst h_c'' corresponds to the threshold field of the periodic transient pattern. Curve *b*, *c* and *d* in figure 1 corresponds to $h \approx h_c''$, $h \approx 1{\cdot}9h_c''$ and $h \approx 2{\cdot}3h_c''$, respectively. The phase diagram of the system is shown in figure 2. Figure 3 shows the easy wavevector q_c versus the magnetic field h for three different values of the extrapolation length. The behaviour of the easy wavevector q_c close to the threshold is evidenced in figure 3 (*b*). For $h \geqslant h_c''$, q_c exhibits the typical dependence which characterizes a second order transition.

The Freederickz threshold field h_c' can be calculated by substituting $q = 0$ into equations (22) and (23). We find

$$\lambda_1 = 0 \tag{24}$$

and

$$\lambda_2^* \operatorname{tg}\left(\frac{\lambda_2^*}{2}\right) = \frac{1}{b}, \tag{25}$$

where λ_2^* is a real number defined as $\lambda_2^* = \lambda_2/i$. For $q = 0$ the growth rate s of equation (21) becomes greater than zero if $h \geqslant h_c' = \lambda_2^*/\pi$. Therefore, by using equation (25) we recover the known Rapini [14] expression for the Freederickz threshold in the presence of finite anchoring

$$\pi h_c' \operatorname{tg}\left(\frac{\pi}{2} h_c'\right) = \frac{1}{b}. \tag{26}$$

To obtain the analytical expression of h_c'' we note that this threshold field corresponds to a change of sign of the curvature of the function $s(q)$ at $q = 0$ (see figure 1). Therefore h_c'' can be calculated by studying the parabolic expansion of $s(q)$ close to

$q = 0$ by using a standard perturbative analysis. We first calculate approximate expressions for λ_1^2 and λ_2^2 at second order in the small parameter q by looking for a solution of equations (22) and (23) of the kind

$$\lambda_1^2(q) = y \tag{27}$$

and

$$\lambda_2^2(q) = -\lambda_2^{*2} + x = -\pi^2 h_c'^2 + x, \tag{28}$$

where x and y are small parameters and λ_2^* is the solution of equation (25). By making a first order expansion of equations (22) and (23) in the small parameters x and y and by exploiting equation (25) we obtain x and y as a function of q^2. By substituting $\lambda_2^2(q)$ into equation (21) and by making a power expansion of $s(q)$ at second order in q we find

$$s(q) = s(0) - \delta(h)q^2 + O(q^4), \tag{29}$$

where

$$\delta(h) = \frac{1}{\pi^2}\left[\frac{K_{33}}{K_{22}} - \frac{\alpha_2^2}{\eta_a\gamma_1}\left(1 - \frac{h^2}{h_c'^2}\right)\left(1 - \frac{1}{[\frac{1}{2} + (\pi^2 h_c'^2 b/4) + (1/4b)]}\right)\right]. \tag{30}$$

The threshold field h_c'' is given by the condition $\delta(h_c'') = 0$ which is satisfied for

$$h_c'' = h_c'\left[\frac{1 - (K_{33}\eta_a\gamma_1)/(K_{22}\alpha_2^2) - 1\Big/\left(\dfrac{1}{2} + \dfrac{\pi^2 h_c'^2 b}{4} + \dfrac{1}{4b}\right)}{1 - 1\Big/\left(\dfrac{1}{2} + \dfrac{\pi^2 h_c'^2 b}{4} + \dfrac{1}{4b}\right)}\right]^{1/2}. \tag{31}$$

In the strong anchoring case ($b = 0$) we recover the known results [2]

$$h_c' = 1, \tag{32}$$

$$h_c'' = \left(1 + \frac{K_{33}\eta_a\gamma_1}{K_{22}\alpha_2^2}\right)^{1/2}, \tag{33}$$

whilst, in the weak anchoring limit ($b \to \infty$), we find

$$h_c' = \frac{1}{\pi}\left(\frac{2}{b}\right)^{1/2}, \tag{34}$$

$$h_c'' = \frac{1}{\pi}\left(\frac{12K_{33}\eta_a\gamma_1}{K_{22}\alpha_2^2}\right)^{1/2}. \tag{35}$$

This means that at large b values the threshold field of the transient pattern is almost independent of the anchoring energy coefficient. Therefore the stability region of the homogeneous twist becomes larger and larger as the anchoring energy coefficient decreases. The typical behaviour of h_c' and h_c'', as given by equations (26) and (31), is shown in figure 2.

4. Conclusions

Here we have investigated the influence of a finite anchoring energy on the characteristic behaviour of the transient patterns which occur in some nematic liquid crystals when a magnetic field is applied above a threshold value. We find that the transition from the homogeneous Freederickz director arrangement to the periodic one always occurs as a second order transition (see figures 1 and 3) and the homogeneous and the transient periodic patterns are present for all values of the anchoring energy coefficient (see the phase diagram in figure 2). This means that the finite anchoring at the two plane surface of the nematic layer affects the qualitative behaviour of the system weakly. An analytical expression for the threshold field of the periodic pattern is obtained as a function of the reduced extrapolation length. In contrast to the threshold of the Freederickz transition, the threshold of the transient pattern does not tend to zero as the anchoring energy coefficient approaches zero.

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